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Note

Injective choosability of planar graphs of girth five and six ☆

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ABSTRACT

An *injective k-coloring* of a graph G is an assignment of k colors to $V(G)$ such that vertices having a common neighbor receive distinct colors. We study the list version of injective colorings of planar graphs. Let $\chi_i^l(G)$ and $\mathrm{mad}(G)$ be the injective choosability number and the maximum average degree of G, respectively. It is proved that (1) for each graph G with $\mathrm{mad}(G) < \frac{10}{3}$, $\chi_i^l(G) \leq \Delta(G) + 4$ if $\Delta(G) \geq 30$ (this conditionally improves some results of Doyon et al. (2010) [9] and Lužar et al. (2009) [11]), $\chi_i^l(G) \leq \Delta(G)+5$ if $\Delta(G) \geq 18$, and $\chi_i^l(G) \leq \Delta(G)+6$ if $\Delta(G) \geq 14$; (2) $\chi_i^l(G) \leq \Delta(G)+2$ if $\mathrm{mad}(G) < 3$ and $\Delta(G) \geq 12$ (this conditionally improves a result of Doyon et al. (2010) [9]).

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1. Introduction

All graphs considered in this paper are simple, finite and undirected. Terminology and notation not defined here are from [1]. Let G be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, Δ and δ) to denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively. The *girth* $g(G)$ of G is the length of a shortest cycle in G. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to v, and let $d(v) = |N(v)|$ denote the degree of v. A k-, k^+- or k^--vertex is a vertex of degree k, at least k or at most k, respectively. A k-vertex v with $N(v) = \{v_1, v_2, \ldots, v_k\}$ such that $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_k)$ is referred to as a $(d(v_1), d(v_2), \ldots, d(v_k))$-vertex.

Recently, the *maximum average degree* is widely used in the study of coloring problems of graphs. It is defined as $\mathrm{mad}\,(G) = \max\{\frac{2|E(H)|}{|V(H)|} : H \text{ is a subgraph of } G\}$. It is easy to see that $\mathrm{mad}\,(G) < \frac{2g}{g-2}$ when G is a planar graph of girth at least g.

A vertex coloring of a graph G is said to be *injective* if any two vertices having a common neighbor get different colors. The *injective chromatic number* of G, denoted by $\chi_i(G)$, is defined to be the minimum number of colors used in injective colorings of G. Clearly, $\chi_i(G) \geq \Delta(G)$ for each graph G. Let $G^{(2)}$ denote the *neighboring graph* of G which is defined by $V(G^{(2)}) = V(G)$ and $E(G^{(2)}) = \{uv : u \text{ and } v \text{ have a common neighbor in } G\}$. In fact, an injective coloring of G is just a classical proper coloring of $G^{(2)}$, and thus $\chi_i(G) = \chi(G^{(2)})$.

A *list assignment* of G is a mapping L that assigns to each vertex v of G a list $L(v)$ of colors. Given a list assignment L of G, an *injective L-coloring* of G is an injective coloring such that each vertex receives a color from its own list. A graph G is *injectively k-choosable* if G has an injective L-coloring for every assignment L with $|L(v)| \geq k$ for every v of G. The *injective choosability number* of G, denoted by $\chi_i^l(G)$, is the smallest integer k such that G is injectively k-choosable. Note that $\chi_i(G) \leq \chi_i^l(G)$ for every graph G.

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In 2002, Hahn et al. [10] introduced and studied the concept of injective coloring, and showed, among other results, that $\chi(G) \le \chi_i(G) \le \Delta(G)(\Delta(G)-1)+1$ for each graph G unless $G = K_2$. Let d and g be positive integers. A planar graph G with $\Delta(G) \ge d$ and $g(G) \ge g$ is referred to as a (d,g)-*graph*. In 2007, Borodin et al. [5] proved that if $(d,g) \in \{(3,24),(4,15),(5,13),(6,12),(7,11),(9,10),(15,8),(30,7)\}$ then each (d,g)-graph G has $\chi_i(G) \le \Delta(G)+1$. Many results of this type appeared in the recent years. Improving a result of [11], Cranston, Kim and Yu proved that $\chi_i(G) = \Delta(G)$ if G is a $(4,13)$-graph. Reducing the maximum degree from 4 to 3, Bu et al. [6] showed that $\chi_i(G) = \Delta(G)$ whenever G is a $(3,20)$-graph. Some other results with the values of $\mathrm{mad}(G)$ or of (d,g) that guarantee a bounded injective chromatic number are given in [2,6–9,11]. In [3], Borodin and Ivanova studied the list version of injective colorings of (d,g)-graphs and proved that $\chi_i(G) = \chi_i^l(G) = \Delta(G)$ if $(d,g) \in \{(16,7),(10,8),(6,10),(5,12)\}$, and that $\chi_i(G) \le \chi_i^l(G) \le \Delta(G)+1$ if $(d,g) = (24,6)$ which improves the result of [2] saying that $\chi_i^l(G) \le \chi^l(G^2) \le \Delta(G)+2$ if G is a $(36,6)$-graph, where $\chi^l(G)$ is the choosability number of G.

In [9], Doyon et al. proved that $\chi_i(G) \le \Delta(G)+3$ if $\mathrm{mad}(G) < \frac{14}{5}$, $\chi_i(G) \le \Delta(G)+4$ if $\mathrm{mad}(G) < 3$, and $\chi_i(G) \le \Delta(G)+8$ if $\mathrm{mad}(G) < \frac{10}{3}$. These results imply that $\chi_i(G) \le \Delta(G)+3$ for $(1,7)$-graphs, $\chi_i(G) \le \Delta(G)+4$ for $(1,6)$-graphs, and $\chi_i(G) \le \Delta(G)+8$ for $(1,5)$-graphs. In [11], it is proved, among other results, that $\chi_i(G) \le \Delta(G)+4$ if G is a $(439,5)$-graph.

Our main results in this note are as follows.

Theorem 1. *Let G be a graph with $\mathrm{mad}(G) < \frac{10}{3}$ and maximum degree Δ. Then $\chi_i^l(G) \le \Delta+4$ if $\Delta \ge 30$, $\chi_i^l(G) \le \Delta+5$ if $\Delta \ge 18$, and $\chi_i^l(G) \le \Delta+6$ if $\Delta \ge 14$.*

Theorem 2. *Let G be a graph with $\Delta(G) \ge 12$. Then $\chi_i^l(G) \le \Delta(G)+2$ if $\mathrm{mad}(G) < 3$.*

Theorems 1 and 2 conditionally improve some results of [9] saying that $\chi_i(G) \le \Delta(G)+8$ if $\mathrm{mad}(G) < \frac{10}{3}$ and that $\chi_i(G) \le \Delta(G)+4$ if $\mathrm{mad}(G) < 3$. Since each planar graph G of girth at least 5 has $\mathrm{mad}(G) < \frac{10}{3}$, and each planar graph G of girth at least 6 has $\mathrm{mad}(G) < 3$, we have the following Corollary 3 which improves a result of [11] saying that $\chi_i(G) \le \Delta(G)+4$ if G is a $(439,5)$-graph, and Corollary 4 which improves the result of [4] on injective coloring saying that $\chi_i(G) \le \chi(G^2) \le \Delta(G)+2$ if G is a $(18,6)$-graph.

Corollary 3. *Let G be a planar graph with $g(G) \ge 5$. Then $\chi_i^l(G) \le \Delta(G)+4$ if $\Delta(G) \ge 30$, $\chi_i^l(G) \le \Delta(G)+5$ if $\Delta(G) \ge 18$, and $\chi_i^l(G) \le \Delta(G)+6$ if $\Delta(G) \ge 14$.*

Corollary 4. *Every planar graph G with $g(G) \ge 6$ and $\Delta(G) \ge 12$ has $\chi_i^l(G) \le \Delta(G)+2$.*

2. Proof of theorems

A graph G is *injectively ℓ-critical* if it is not injectively ℓ-choosable but all its proper subgraphs are injectively ℓ-choosable. An *h-thread* of G is an induced path of length $h+1$ of which each internal vertex has degree 2 in G. Before proving Theorems 1 and 2, we need the following lemmas on the structure of injectively ℓ-critical graphs. The first lemma was first implicitly used in [3]. We give a proof here for completeness.

Lemma 5 ([3]). *Let G be an injectively ℓ-critical graph. If $\ell > \Delta$, then $\delta(G) \ge 2$ and G has no 2-thread.*

Proof. Let L be an arbitrary list assignment of G with $|L(v)| = \ell > \Delta$ for each v of G.

First, we show that $\delta(G) \ge 2$. Assume to the contrary that u is a 1-vertex adjacent to v in G. By the choice of G, $G-u$ has an injective L-coloring c. Since $d_{G^{(2)}}(u) = d(v)-1 \le \Delta-1 < \ell$, there is at least one color not used on $N_{G^{(2)}}(u)$ with respect to c. So, c can be extended to an injective L-coloring of G, contradicting the criticality of G.

Now, we show that G has no 2-thread. If it is not true, let v_1 and v_2 be two adjacent 2-vertices of G, and let u and w be the other neighbors of v_1 and v_2, respectively. By the choice of G, the graph $G-v_1-v_2$ admits an injective L-coloring c'. Since $d_{G^{(2)}}(v_1) = d(u)+d(v_2)-2 \le \Delta$ and $d_{G^{(2)}}(v_2) = d(v_1)+d(w)-2 \le \Delta$, $L(v_i)$ has a color not used on $N_{G^{(2)}}(v_i)$ with respect to c', $i=1,2$. So, c' can be extended to an injective L-coloring of G, contradicting the criticality of G. □

Lemma 6. *Let G be an injectively ℓ-critical graph. Suppose that v is a 2-vertex of G with $N(v) = \{v_1, v_2\}$. If $d(v_1)+d(v_2) \le \ell+1$, then $\sum_{u\in N(v_i)} d(u) \ge \ell + d(v_i)$ for $i = 1, 2$.*

Proof. If it is not true, we may suppose by symmetry that $d(v_1)+d(v_2) \le \ell+1$ and $\sum_{u\in N(v_1)} d(u) < \ell + d(v_1)$. Let L be an arbitrary list assignment of G with $|L(v)| \ge \ell$ for each vertex v of G. Then the graph $G - vv_1$ has an injective L-coloring c' by the criticality of G. Remove the colors on v and v_1. Since $d(v_1)+d(v_2)-2 \le \ell-1$ (resp. $\sum_{u\in N(v_1)} d(u) - d(v_1) < \ell$), at most $\ell-1$ colors are used by c' on the vertices in $N_{G^{(2)}}(v)$ (resp. $N_{G^{(2)}}(v_1)$), thus there are still colors in $L(v)$ (resp. $L(v_1)$) available for v (resp. v_1). Therefore, G is injectively L-colorable, a contradiction to the choice of G. □

Lemma 7. *Let G be an injectively ℓ-critical graph. Suppose that v and v_1 are adjacent 3-vertices with $N(v) = \{v_1, v_2, v_3\}$ and $N(v_1) = \{v, u, w\}$ (see Fig. 1). Then, either $d(u)+d(w) \ge \ell$, or $d(v_2)+d(v_3) \ge \ell$.*

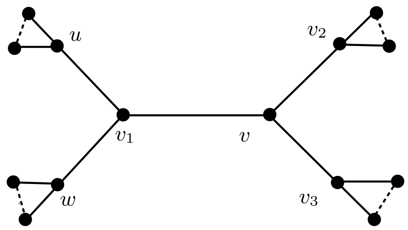

Fig. 1. Two adjacent 3-vertices.

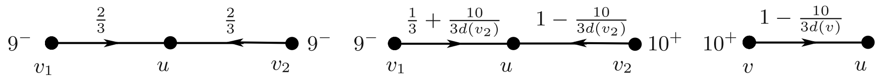

Fig. 2. Weight transferred to a 2-vertex u.

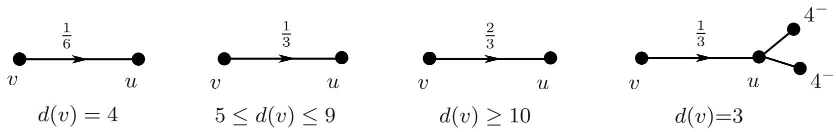

Fig. 3. Weight transferred from v to a 3-vertex u.

Proof. Assume to the contrary that $d(v_2) + d(v_3) < \ell$ and $d(u) + d(w) < \ell$. Let L be an arbitrary list assignment of G with $|L(v)| \geq \ell$ for each v of G. By the choice of G, the graph $G - vv_1$ admits an injective L-coloring c'. Remove the colors of v and v_1. Since $d_{G^{(2)}}(v) = d(v_2) + d(v_3) - 2 + 2 \leq \ell - 1$ and $d_{G^{(2)}}(v_1) = d(u) + d(w) - 2 + 2 \leq \ell - 1$, $L(v)$ and $L(v_1)$ have a color not used on $N_{G^{(2)}}(v)$ and $N_{G^{(2)}}(v_1)$, respectively, with respect to c'. So, G is injectively L-colorable that contradicts the criticality of G. With the similar argument, one can prove the cases when $u = v_2$ or $w = v_3$. $\square$

We proceed to prove our theorems.

Proof of Theorem 1. Let $\ell = \Delta + 4$ if $\Delta \geq 30$, $\ell = \Delta + 5$ if $\Delta \geq 18$, and $\ell = \Delta + 6$ if $\Delta \geq 14$. If the theorem is not true, there must exist an injectively ℓ-critical graph. Let G be such a graph, and let $|V(G)| = n$. We will use a discharging method. The initial charge is defined as $\omega(v) = d(v)$ for each vertex v of G. Then, $\sum_{v\in V(G)} \omega(v) < \frac{10n}{3}$ since $\mathrm{mad}(G) < \frac{10}{3}$. We apply the following rules to redistribute the weight that leads to a new charge $\omega'(v)$, and will show that $\omega'(v) \geq \frac{10}{3}$ for each vertex v of G. This contradiction completes the proof. We transfer charges as follows.

(R1) Each k-vertex v to every adjacent 2-vertex u, transfers (see Fig. 2)
- (R1.1) $\frac{2}{3}$, if $3 \leq k \leq 9$ and u is a $(9^-, 9^-)$-vertex;
- (R1.2) $\frac{1}{3} + \frac{10}{3k'}$, if $3 \leq k \leq 9$ and u is a (k, k')-vertex with some $k' \geq 10$;
- (R1.3) $1 - \frac{10}{3k}$, if $k \geq 10$.

(R2) Each k-vertex v to every adjacent 3-vertex u, transfers (see Fig. 3)
- (R2.1) $\frac{1}{6}$, if $k = 4$;
- (R2.2) $\frac{1}{3}$, if $5 \leq k \leq 9$;
- (R2.3) $\frac{2}{3}$, if $k \geq 10$;
- (R2.4) $\frac{1}{3}$, if $k = 3$ and u is a $(3, 4^-, 4^-)$-vertex.

(R3) Each 4-vertex receives $\frac{1}{3}$ from each of its adjacent k-vertices for $5 \leq k \leq 9$.

(R4) Each 5-vertex receives $\frac{1}{3}$ from each of its adjacent k-vertices for $6 \leq k \leq 9$.

(R5) Each k-vertex for $4 \leq k \leq 9$ receives $\frac{2}{3}$ from each of its adjacent 10^+-vertices.

Note that for $k \in \{3, \ldots, 9\}$, each k-vertex transfers at most $\frac{2}{3}$ to each of its adjacent 2-vertices by (R1.1) and (R1.2).

Let v be a k-vertex with $N(v) = \{v_1, v_2, \ldots, v_k\}$ such that $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_k)$. By Lemma 5, we have $k \geq 2$, and no 2-vertices are adjacent in G.

If $k \geq 10$, then $1 - \frac{10}{3k} \geq \frac{2}{3}$, v transfers at most $1 - \frac{10}{3k}$ to each of its neighbors by (R1.3) and (R2.3), and hence $\omega'(v) \geq k - k \times (1 - \frac{10}{3k}) = \frac{10}{3}$.

We will show in the following Cases 1 to 4 that $\ell \geq \Delta + 4$ and $\Delta \geq 14$ suffice for guaranteeing $\omega'(v) \geq \frac{10}{3}$ while $2 \leq d(v) \leq 5$. Then, we distinguish three possibilities according to the values of ℓ and Δ to compute $\omega'(v)$ in Case 5 for vertices of degree between 6 and 9 separately. For convenience, we use v_1' to denote the neighbor of v_1 other than v if $d(v_1) = 2$.

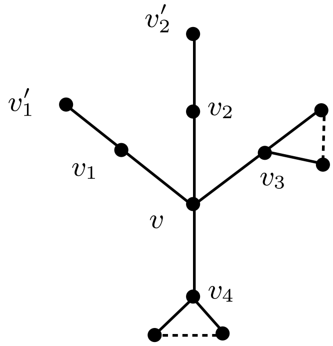

Fig. 4. 4-vertex adjacent two 2-vertices.

Note that in the following Cases 1 to 4, we suppose that $\ell \geq \Delta + 4$ and $\Delta \geq 14$.

Case 1. $k = 2$. If v is a $(9^-, 9^-)$-vertex, then by (R1.1) each 9^--vertex sends $\frac{2}{3}$ to v, thus $\omega'(v) \geq 2 + 2 \times \frac{2}{3} = \frac{10}{3}$. If v is a $(9^-, 10^+)$-vertex, then $\omega'(v) \geq 2 + (\frac{1}{3} + \frac{10}{3d(v_2)}) + (1 - \frac{10}{3d(v_2)}) = \frac{10}{3}$ by (R1.2). Otherwise, v is a $(10^+, 10^+)$-vertex and hence $\omega'(v) \geq 2 + (1 - \frac{10}{3d(v_1)}) + (1 - \frac{10}{3d(v_2)}) \geq \frac{10}{3}$ by (R1.3).

Case 2. $k = 3$. If $d(v_1) \geq 4$, then v receives at least $\frac{1}{6}$ from each of its neighbors by (R2), and hence $\omega'(v) \geq 3 + 3 \times \frac{1}{6} \geq \frac{10}{3}$. We suppose that $d(v_1) = 2$ or 3.

First suppose that v_1 is a 2-vertex which receives at most $\frac{2}{3}$ by (R1). Since $d(v) + d(v_1') \leq \Delta + 3 < \ell$, we have $d(v_2) + d(v_3) \geq \Delta + 5 \geq 19$ by Lemma 6 (since $\Delta \geq 14$), and then v_2 is a 5^+-vertex sending to v at least $\frac{1}{3}$ and v_3 is a 10^+-vertex sending to v at least $\frac{2}{3}$ by (R2). So, $\omega'(v) \geq 3 - \frac{2}{3} + 1 = \frac{10}{3}$.

Now, suppose that $d(v_1) = 3$. Since $d(v_1) + d(v_2) \leq \Delta + 3$ and $d(v_1) + d(v_3) \leq \Delta + 3$, neither v_2 nor v_3 can be a $(3, 4^-, 4^-)$-vertex by Lemma 7. If $d(v_2) + d(v_3) \geq \Delta + 4$, then $d(v_2) \geq 4$ and either both v_2 and v_3 are 5^+-vertices or v_3 is a 10^+-vertex, v receives totally at least $\frac{2}{3}$ from v_2 and v_3 and sends at most $\frac{1}{3}$ to v_1 by (R2), and thus $\omega'(v) \geq 3 + \frac{2}{3} - \frac{1}{3} = \frac{10}{3}$. Otherwise, we have that $d(v_2) + d(v_3) < \Delta + 4$. Now $d(u) + d(w) \geq \Delta + 4$ (see Fig. 1) by Lemma 7 where both u and w are neighbors of v_1, and v sends no charge to v_1 by (R2.4) since v_1 cannot be a $(3, 4^-, 4^-)$-vertex. If $d(v_3) \leq 4$, then v is a $(3, 4^-, 4^-)$-vertex who receives $\frac{1}{3}$ from v_1 and receives at least $\frac{1}{6}$ from each of v_2 and v_3 by (R2), and hence $\omega'(v) \geq 3 + \frac{2}{3} > \frac{10}{3}$. Suppose that $d(v_3) \geq 5$. Now, v_3 sends at least $\frac{1}{3}$ to v by (R2.2) and (R2.3), v_2 receives no charge from v since v_2 cannot be a $(3, 4^-, 4^-)$-vertex, and hence $\omega'(v) \geq 3 + \frac{1}{3} = \frac{10}{3}$.

Case 3. $k = 4$. If $d(v_1) \geq 3$, then v sends at most $\frac{1}{6}$ to each adjacent vertex by (R2.1), and $\omega'(v) \geq 4 - 4 \times \frac{1}{6} = \frac{10}{3}$. Suppose that $d(v_1) = 2$ (see Fig. 4). Now $\sum_{u \in N(v)} d(u) \geq \Delta + 8$ by Lemma 6 that yields $d(v_2) + d(v_3) + d(v_4) \geq \Delta + 6 \geq 20$ (since $\Delta \geq 14$). Either $d(v_4) \geq 10$ and v receives $\frac{2}{3}$ from v_4 by (R5), or $d(v_4) \geq d(v_3) \geq$ and v receives totally at least $\frac{2}{3}$ from v_3 and v_4 by (R3). We distinguish three possibilities depending on $d(v_2)$. If $d(v_2) = 2$, then $d(v_3) \geq 4$, v sends at most $\frac{2}{3}$ to each of v_1 and v_2 by (R1.1) and (R1.2), and so $\omega'(v) \geq 4 - 2 \times \frac{2}{3} + \frac{2}{3} = \frac{10}{3}$. If $d(v_2) = 3$, then $d(v_3) \geq 3$, v sends totally at most $\frac{1}{3}$ to v_2 and v_3 by (R2.1), and $\omega'(v) \geq 4 + \frac{2}{3} - \frac{2}{3} - \frac{1}{3} = \frac{11}{3}$. Otherwise, we have $d(v_2) \geq 4$, and then $\omega'(v) \geq 4 + \frac{2}{3} - \frac{2}{3} > \frac{10}{3}$.

Case 4. $k = 5$. If $d(v_1) \geq 3$, then each neighbor receives at most $\frac{1}{3}$ from v by (R2.2) and (R3), $\omega'(v) \geq 5 - 5 \times \frac{1}{3} = \frac{10}{3}$. So, we suppose that $d(v_1) = 2$. Now $d(v) + d(v_1') \leq \Delta + 5 \leq \ell + 1$, thus $\sum_{i=2}^{5} d(v_i) \geq \Delta + 7 \geq 21$ by Lemma 6 (since $\Delta \geq 14$). So, v is adjacent to at most three 2-vertices since otherwise $\sum_{i=2}^{5} d(v_i) \leq \Delta + 6$, and v_5 is a 6^+-vertex who sends $\frac{1}{3}$ to v by (R4). If $d(v_3) = 2$, then v is either a $(2, 2, 2, 3^+, 10^+)$-vertex or a $(2, 2, 2, 8^+, 9)$-vertex, it receives totally at least $\frac{2}{3}$ from v_4 and v_5 by (R4) and (R5) in both cases, and sends at most $\frac{1}{3}$ to v_4 by (R2.2) and (R3), and hence $\omega'(v) \geq 5 - 3 \times \frac{2}{3} - \frac{1}{3} + \frac{2}{3} = \frac{10}{3}$. If $d(v_2) = 2$ and $d(v_3) \geq 3$, then v is a $(2, 2, 3^+, 3^+, 7^+)$-vertex who receives $\frac{1}{3}$ from v_5 by (R4) and sends out totally at most $\frac{2}{3}$ to v_3 and v_4 by (R2.2) and (R3), and hence $\omega'(v) \geq 5 - 3 \times \frac{2}{3} + \frac{1}{3} = \frac{10}{3}$. Otherwise, we have $d(v_2) \geq 3$, and so $\omega'(v) \geq 5 - \frac{2}{3} - 3 \times \frac{1}{3} + \frac{1}{3} > \frac{10}{3}$.

Case 5. Finally, we suppose that $6 \leq k \leq 9$. If $d(v_1) \geq 3$, then each neighbor receives at most $\frac{1}{3}$ from v by (R2.2), (R3) and (R4), and thus $\omega'(v) \geq k - k \times \frac{1}{3} \geq 4 > \frac{10}{3}$. Suppose that $d(v_1) = 2$. We discuss three situations depending on the values of ℓ and Δ.

We first consider that $\ell = \Delta + 4$ and $\Delta \geq 30$. Note that $\Delta + 6 - k > 10$ since $\Delta \geq 30$ and $k \leq 9$. If each 2-vertex in $N(v)$ has a $(\Delta + 6 - k)^+$-vertex as a neighbor, then v sends to each neighbor at most $(\frac{1}{3} + \frac{10}{3(\Delta+6-k)})$ by the discharging rules, and hence $\omega'(v) \geq k - k \times (\frac{1}{3} + \frac{10}{3} \times \frac{1}{\Delta+6-k}) \geq \frac{2k}{3} - \frac{10}{3} \times \frac{k}{36-k} \geq \frac{10}{3}$ since $\Delta \geq 30$. So, we further suppose, without loss of generality, that $d(v_1') \leq \Delta + 5 - k$. Now, $\sum_{i=2}^{k} d(v_i) \geq \Delta + 2 + k$ by Lemma 6 since $d(v) + d(v_1') \leq \Delta + 5 = \ell + 1$. If $k \in \{6, 7\}$, then either $d(v_k) \geq 10$ that indicates v receives $\frac{2}{3}$ from v_k by (R5) and thus $\omega'(v) \geq k - (k-1) \times \frac{2}{3} + \frac{2}{3} \geq \frac{10}{3}$, or

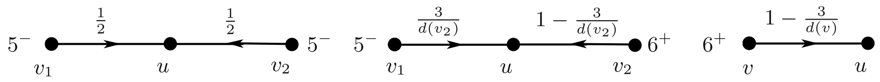

Fig. 5. Weight transferred to a 2-vertex u.

$d(v_{k-1}) \geq 6$ that indicates both v_{k-1} and v_k receive no charge from v and thus $\omega'(v) \geq k-(k-2)\times\frac{2}{3} \geq \frac{10}{3}$. If $k \in \{8, 9\}$, then v_k is a 6^+-vertex who receives no charge from v, and hence $\omega'(v) \geq k-(k-1)\times\frac{2}{3} \geq \frac{10}{3}$.

Now, we consider the situation that $\ell = \Delta+5$ and $\Delta \geq 18$. If $k = 6$ and $d(v_3) \geq 3$, then each neighbor but v_1 and v_2 receives at most $\frac{1}{3}$ from v, and hence $\omega'(v) \geq 6-2\times\frac{2}{3}-4\times\frac{1}{3} = \frac{10}{3}$. Suppose that $k = 6$ and $d(v_3) = 2$. Note that $\sum_{i=2}^{6} d(v_i) \geq \Delta+9 \geq 27$ by Lemma 6 since $d(v)+d(v_1') \leq \Delta+6 = \ell+1$. Either v is a $(2, 2, 2, 2^+, 3^+, 10^+)$-vertex, or v is a $(2, 2, 2, 5^+, 7^+, 9^-)$-vertex. In the former case, v receives at least $\frac{2}{3}$ from v_6 by (R5), sends at most $\frac{1}{3}$ to v_5 by (R2.2), (R3) and (R4), and hence $\omega'(v) \geq 6-4\times\frac{2}{3}-\frac{1}{3}+\frac{2}{3} = \frac{11}{3}$. In the latter, we have $\omega'(v) \geq 6-3\times\frac{2}{3}-\frac{1}{3} = \frac{11}{3}$.

Suppose that $k \in \{7, 8, 9\}$. If each 2-vertex in $N(v)$ has a $(\Delta+7-k)^+$-vertex as a neighbor, then v sends to each neighbor at most $(\frac{1}{3}+\frac{10}{3(\Delta+7-k)})$ by the discharging rules (note that $\Delta+7-k > 10$), and hence $\omega'(v) \geq k-k\times(\frac{1}{3}+\frac{10}{3}\times\frac{1}{\Delta+7-k}) \geq \frac{2k}{3}-\frac{10}{3}\times\frac{k}{25-k} \geq \frac{10}{3}$ since $\Delta \geq 18$. So, we suppose, without loss of generality, that $d(v_1') \leq \Delta+6-k$. Now, $\sum_{i=2}^{k} d(v_i) \geq \Delta+3+k$ by Lemma 6. If $k = 7$ and $d(v_5) \geq 3$, then $\omega'(v) \geq 7-4\times\frac{2}{3}-3\times\frac{1}{3} = \frac{10}{3}$. If $k = 7$ and $d(v_5) = 2$, then v_7 is a 10^+-vertex (since $\Delta \geq 18$) and sends $\frac{2}{3}$ to v by (R5), and thus $\omega'(v) \geq 7-6\times\frac{2}{3}+\frac{2}{3} = \frac{11}{3}$. If $8 \leq k \leq 9$ and $d(v_{k-1}) = 2$, then $d(v_k) \geq 16$, thus $\omega'(v) \geq k-(k-1)\times\frac{2}{3}+\frac{2}{3} \geq 4$. Otherwise, we have $8 \leq k \leq 9$ and $d(v_{k-1}) \geq 3$, and then $\omega'(v) \geq k-(k-2)\times\frac{2}{3}-2\times\frac{1}{3} \geq \frac{10}{3}$.

The only remaining situation is that $\ell = \Delta+6$ and $\Delta \geq 14$. If $k = 6$ and $d(v_3) \geq 3$, then each neighbor but v_1 and v_2 receives at most $\frac{1}{3}$ from v, and hence $\omega'(v) \geq 6-2\times\frac{2}{3}-4\times\frac{1}{3} = \frac{10}{3}$. Suppose that $k = 6$ and $d(v_3) = 2$. Since $d(v_1')+d(v) \leq \Delta+6 < \ell+1$, $\sum_{i=2}^{6} d(v_i) \geq \Delta+10 \geq 24$ by Lemma 6. Note that $\Delta \geq 14$, v_6 is a 7^+-vertex and receives no charge from v. If $d(v_4) \geq 3$, then v is a $(2, 2, 2, 3^+, 3^+, 7^+)$-vertex who sends totally at most $\frac{2}{3}$ to v_4 and v_5 by (R2.2), (R3) and (R4), and thus $\omega'(v) \geq 6-3\times\frac{2}{3}-2\times\frac{1}{3} = \frac{10}{3}$. Otherwise, we have $d(v_4) = 2$, and hence v is either a $(2, 2, 2, 2, 4^+, 10^+)$-vertex or a $(2, 2, 2, 2, 9, 9)$-vertex. In the former case, v receives at least $\frac{2}{3}$ from v_6 by (R5), sends at most $\frac{1}{3}$ to v_5 by (R3) and (R4), and hence $\omega'(v) \geq 6-4\times\frac{2}{3}-\frac{1}{3}+\frac{2}{3} = \frac{11}{3}$. In the latter, we have $\omega'(v) \geq 6-4\times\frac{2}{3} = \frac{10}{3}$.

If $k = 7$ and $d(v_5) \geq 3$, then $\omega'(v) \geq 7-4\times\frac{2}{3}-3\times\frac{1}{3} \geq \frac{10}{3}$. Suppose that $k = 7$ and $d(v_5) = 2$. By Lemma 6, we have $\sum_{i=2}^{7} d(v_i) \geq \Delta+11 \geq 25$ since $d(v)+d(v_1') \leq \Delta+7 = \ell+1$. Then, either v_7 is a 10^+-vertex who sends $\frac{2}{3}$ to v by (R5), or both v_6 and v_7 are 8^+-vertices who receive no charge from v. In both cases, $\omega'(v) \geq 7-5\times\frac{2}{3} = \frac{11}{3}$.

Suppose that $k \in \{8, 9\}$. If each 2-vertex in $N(v)$ has a $(\Delta+8-k)$-vertex as a neighbor, then v sends to each neighbor at most $(\frac{1}{3}+\frac{10}{3(\Delta+8-k)})$ by the discharging rules since $\Delta+8-k > 10$, and thus $\omega'(v) \geq k-k\times(\frac{1}{3}+\frac{10}{3}\times\frac{1}{\Delta+8-k}) \geq \frac{2k}{3}-\frac{10}{3}\times\frac{k}{22-k} \geq \frac{10}{3}$ since $\Delta \geq 14$. Without loss of generality, we suppose that $d(v_1') \leq \Delta+7-k$. Now $\sum_{i=2}^{k} d(v_i) \geq \Delta+4+k$ by Lemma 6 since $d(v)+d(v_1') \leq \Delta+7 = \ell+1$. If $d(v_{k-1}) = 2$ then v_k is a 13^+-vertex who sends $\frac{2}{3}$ to v by (R1.3), and thus $\omega'(v) \geq k-(k-1)\times\frac{2}{3}+\frac{2}{3} \geq 4$. Otherwise we have $d(v_{k-1}) \geq 3$, and then $\omega'(v) \geq k-(k-2)\times\frac{2}{3}-2\times\frac{1}{3} \geq \frac{10}{3}$.

In all the cases, we have $\omega'(v) \geq \frac{10}{3}$ for every vertex v of G, which contradicts $\mathrm{mad}(G) < \frac{10}{3}$. This contradiction totally completes the proof of Theorem 1. $\square$

Proof of Theorem 2. Let $\Delta(G) = \Delta$. As in the proof of Theorem 1, we suppose that G is an injective $(\Delta+2)$-critical graph and take the discharging method. The initial charge is still $\omega(v) = d(v)$ for each vertex v of G. Following the discharging rules below, we will get a new charge $\omega'(v) \geq 3$ for each vertex v, which contradicts $\mathrm{mad}(G) < 3$.

(R1) Each k-vertex v to each of its adjacent 2-vertices u, transfers (see Fig. 5)
  (R1.1) $\frac{1}{2}$, if $3 \leq k \leq 5$ and u is a $(k, 5^-)$-vertex;
  (R1.2) $\frac{3}{k'}$, if $3 \leq k \leq 5$ and u is a (k, k')-vertex with some $k' \geq 6$;
  (R1.3) $1-\frac{3}{k}$, if $k \geq 6$.
(R2) Each 6^+-vertex sends $\frac{1}{2}$ to its neighbor v if $d(v) = 3$ or 4.

Let v be a k-vertex with $N(v) = \{v_1, v_2, \ldots, v_k\}$ such that $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_k)$.

If $k \geq 6$, then $1-\frac{3}{k} \geq \frac{1}{2}$, and v transfers at most $1-\frac{3}{k}$ to each of its neighbors by (R1.3) and (R2), and hence $\omega'(v) \geq k-k\times(1-\frac{3}{k}) = 3$. If v is a $(5^-, 5^-)$-vertex, then by (R1.1) v receives $\frac{1}{2}$ from each of its neighbors, and thus $\omega'(v) \geq 2+2\times\frac{1}{2} \geq 3$. If v is a $(5^-, 6^+)$-vertex, then $\omega'(v) \geq 2+\frac{3}{d(v_2)}+(1-\frac{3}{d(v_2)}) = 3$ by (R1.2). If v is a $(6^+, 6^+)$-vertex then $\omega'(v) \geq 2+(1-\frac{3}{d(v_1)})+(1-\frac{3}{d(v_2)}) \geq 3$ by (R1.3).

Suppose that $k \in \{3, 4, 5\}$. If $d(v_1) \geq 3$, then $\omega'(v) \geq d(v) \geq 3$ since v sends no charge to its neighbors. So, we further suppose that $d(v_1) = 2$.

First, we consider that $k = 3$. Since $\Delta \geq 12$, we have $d(v_2)+d(v_3) \geq \Delta+3 \geq 15$ by Lemma 6, and thus v is a $(2, 3^+, 8^+)$-vertex that receives $\frac{1}{2}$ from v_3 by (R2) and sends at most $\frac{1}{2}$ to v_1 by (R1). So, $\omega'(v) \geq 3-\frac{1}{2}+\frac{1}{2} \geq 3$.

Second, we consider that $k = 4$. If each 2-vertex in $N(v)$ has a Δ-vertex as a neighbor, then v sends at most $\frac{3}{\Delta}$ to each neighbor, and $\omega'(v) \geq 4 - 4 \times \frac{3}{\Delta} \geq 4 - 4 \times \frac{3}{12} = 3$. Without loss of generality, suppose that the neighbor of v_1 other than v has degree at most $\Delta - 1$. Now $d(v_2) + d(v_3) + d(v_4) \geq \Delta + 4$ by Lemma 6. Since $\Delta \geq 12$, v_4 is a 6^+-vertex who sends $\frac{1}{2}$ to v by (R2). Since v sends at most $\frac{1}{2}$ to each of v_2 and v_3, $\omega'(v) \geq 4 - 3 \times \frac{1}{2} + \frac{1}{2} = 3$.

Finally, we consider the situation that $k = 5$. If each 2-vertex in $N(v)$ has a $(\Delta - 1)^+$-vertex as a neighbor, then by discharging rules, v sends at most $\frac{3}{\Delta-1}$ to each neighbor, and thus $\omega'(v) \geq 5 - 5 \times \frac{3}{\Delta-1} \geq 5 - 5 \times \frac{3}{11} > 3$. So, we suppose that the neighbor of v_1 other than v has degree at most $\Delta - 2$. By Lemma 6, $\sum_{i=2}^{5} v_i \geq \Delta + 5 \geq 17$, and therefore v_5 is a 5^+-vertex who receives no charge from v. Now $\omega'(v) \geq 5 - 4 \times \frac{1}{2} = 3$. This concludes the proof. □

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